# DIRICHLET DYNAMICAL ZETA FUNCTION FOR BILLIARD FLOW

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Abstract. We study the Dirichlet dynamical zeta function $\eta_D(s)$ for billiard flow corresponding to several strictly convex disjoint obstacles. For large $\operatorname{Re} s$ we have $\eta_D(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, $a_n \in \mathbb{R}$ and $\eta_D$ admits a meromorphic continuation to $\mathbb{C}$. We obtain some conditions of the frequencies $\lambda_n$ and some sums of coefficients $a_n$ which imply that $\eta_D$ cannot be prolonged as entire function.

## 1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with $C^\infty$ smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every $D_j$ has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary of $D$. We consider the (non-grazing) billiard flow $\varphi_t$ (see [Pet, Section 2] for the definition). Next the periodic trajectories will be called periodic rays. For any periodic ray $\gamma$, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of $\gamma$ at the obstacles. Denote by $P_\gamma$ the associated linearized Poincaré map (see [PS17, Section 2.3] for the definition).

Let $\mathcal{P}$ be the set of all oriented periodic rays. The counting function of the lengths of primitive periodic rays $\Pi$ satisfies

$$\sharp\{\gamma \in \Pi :\ \tau^\sharp(\gamma) \leqslant x\} \sim \frac{\mathrm{e}^{hx}}{hx}, \quad x \to +\infty, \tag{1.2}$$

for some $h > 0$ (see for instance, [PP90, Theorem 6.5] for weak-mixing suspension symbolic flows). Thus there exists an infinite number of primitive periodic trajectories and for every small $\epsilon > 0$ we have the estimate

$$e^{(h-\epsilon)x} \leq \sharp\{\gamma \in \mathcal{P} :\ \tau(\gamma) \leqslant x\} \leqslant \mathrm{e}^{(h+\epsilon)x},\ x > C_\epsilon. \tag{1.3}$$

Moreover, for some positive constants $C_1, d_1, d_2$ we have (see for instance [Pet99, Appendix])

$$C_1 \mathrm{e}^{d_1 \tau(\gamma)} \leqslant |\det(\mathrm{Id} - P_\gamma)| \leqslant \mathrm{e}^{d_2 \tau(\gamma)}, \quad \gamma \in \mathcal{P}. \tag{1.4}$$

By using these estimates, we define for $\mathrm{Re}(s) \gg 1$ the Dirichlet dynamical zeta function $\eta_D(s)$ by

$$\eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where the sums run over all oriented periodic rays. This zeta function is important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d \setminus \bar{D}$ with Dirichlet boundary conditions on $\partial D$ (see [CP22, §1] for more details). Denote by $\sigma_a \in \mathbb{R}, \sigma_c \in \mathbb{R}$ the abscissa of absolute convergence and the abscissa of convergence of $\eta_D$, respectively.

It was proved in [CP22, Theorem 1 and Theorem 4] that $\eta_D$ admits a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. On the other hand, for $d = 2$ [Sto01] and for $d \geqslant 3$ under some conditions [Sto12] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant \sigma_a - \varepsilon$.

There is a conjecture that $\eta_D$ cannot be prolonged as *entire function*. This conjecture was established for obstacles with real analytic boundary (see [CP22, Theorem 3]) and for obstacles with sufficiently small diameters [Ika90b], [Sto09] and $C^\infty$ smooth boundary. If $\eta_D(s)$ is not an entire function, then we obtain two important corollaries:

(i) $\eta_D$ has infinite number of poles in some strip $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq \beta\}$ (see [Pet, Section 3] for a lower bound of the counting function of poles),

(ii) The modified Lax-Phillips conjecture (MLPC) for scattering resonances introduced by Ikawa [Ika90a] holds. (MLPC) says that there exists a strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z \leq \alpha\}$ containing an infinite number of scattering resonances for Dirichlet Laplacian in $\mathbb{R}^d \setminus \bar{D}$ (see [CP22, Section 1] for definitions and more precise results).

Let $\rho \in C_0^\infty(\mathbb{R}; \mathbb{R}_+)$ be an even function with $\mathrm{supp}\, \rho \subset [-1, 1]$ such that $\rho(t) > 1 \quad \text{if} \quad |t| \leqslant 1/2$.

Let $(\ell_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be sequences of positive numbers such that $\ell_j \geqslant d_0 = \min_{k \neq m} \mathrm{dist}\,(D_k, D_m) > 0$, $m_j \geqslant \max\{1, \frac{1}{d_0}\}$ and let $\ell_j \to \infty$, $m_j \to \infty$ as $j \to \infty$. Set $\rho_j(t) = \rho(m_j(t - \ell_j)), \quad t \in \mathbb{R}$, and introduce the distribution $\mathcal{F}_{\mathrm{D}}(t) \in \mathcal{S}'(\mathbb{R}^+)$ by

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma) \delta(t - \tau(\gamma))}{|\det(I - P_\gamma)|^{1/2}}.$$

We have the following

**Proposition 1.1.** *The function $\eta_{\rm D}(s)$ cannot be prolonged as an entire function of $s$ if and only if there exists $\alpha_0 > 0$ such that for any $\beta > \alpha_0$ we can find sequences $(\ell_j), (m_j)$ with $\ell_j \nearrow \infty$ as $j \to \infty$ such that for all $j \geqslant 0$ one has ${\rm e}^{\beta \ell_j} \leqslant m_j \leqslant {\rm e}^{2\beta\ell_j}$ and*

$$|\langle \mathcal{F}_{\rm D}, \rho_j \rangle| \geqslant {\rm e}^{-\alpha_0 \ell_j}. \tag{1.5}$$

More precisely, if $\eta_D$ cannot be prolonged as entire function, the existence of sequences $(\ell_j)$, $(m_j)$ with the above properties has been proved by Ikawa [Ika90a, Prop.2.3], while in the proof of Theorem 1.1 in [Pet] it was established that if such sequences exist, the function $\eta_D$ has an infinite number of poles.

The conditions of Proposition 1.1 are difficult to verify. The purpose of this Note is to find another conditions which imply that $\eta_D$ cannot be prolonged as entire function. For this purpose we exploit the local trace formula (see [Pet, Theorem 2.1]) and the summability by typical means of Dirichlet series introduced by Hardy and Riesz [HR64] (see also [DS22, Section 2]). It is convenient to write $\eta_D(s)$ as a Dirichlet series

$$\eta_D(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \ {\rm Re}\, s \gg 1, \tag{1.6}$$

where the frequencies are arranged as follows

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$$

and

$$a_n = \sum_{\gamma \in \mathcal{P}, \tau(\gamma) = \lambda_n} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma)}{|\det({\rm Id} - P_\gamma)|^{1/2}}. \tag{1.7}$$

Our main result is the following

**Theorem 1.1.** *Suppose $\sigma_c < 0$. Assume that there exist constants $C > 0$, $\delta > h + 1$, $-\gamma < \sigma_c$ and an increasing sequence $m_j \nearrow \infty$ such that*

$$\lambda_{m_j} - \lambda_{m_j - 1} \geq C e^{-\delta \lambda_{m_j}}, \tag{1.8}$$

$$|\sum_{n \geq m_j} a_n| \geq e^{-\gamma \lambda_{m_j}}. \tag{1.9}$$

*Then $\eta_D(s)$ cannot be prolonged as entire function.*

The condition $\sigma_c < 0$ is not a restriction since if $\sigma_c \geq 0$, the Dirichlet series

$$\eta_D(s + \sigma_c + 1) = \sum_n (a_n e^{-\lambda_n(\sigma_c + 1)}) e^{-\lambda_n s} = \sum_n b_n e^{-\lambda_n s}$$

is convergent for $\operatorname{Re} s > -1$, hence it has a negative abscissa of convergence $\sigma_b$ and $\eta_D(s + \sigma_c + 1)$ is entire if and only if $\eta_D(s)$ is entire. Moreover, in the proof of Theorem 1.1 (see section 3), assuming $\eta_D$ entire, one has the property

$$\forall A < \sigma_c,\ \exists C_A > 0,\ \ |\eta_D(s)| \leq C_A(1 + |\operatorname{Im} s|),\ \operatorname{Re} s \geq A$$

which is satisfied also for $\eta_D(s + \sigma_c + 1)$ with another constants $B_A$. Thus we may apply Theorem 1.1 if instead of (1.9) one has the estimate

$$|\sum_{n \geq m_j} b_n| \geq e^{-\gamma_1 \lambda_{m_j}},\ -\gamma_1 < \sigma_b. \tag{1.10}$$

The assumptions on $\lambda_n$ and $a_n$ in Theorem 1.1 are satisfied if *Bohr condition* (see for instance, [DS22, §3.13])

$$(BC) \qquad \exists C_1 > 0,\ \exists \ell > 0,\ \ \forall n > 0,\ \lambda_{n+1} - \lambda_n \geq C_1 e^{-\ell \lambda_n}$$

holds. Indeed, it is well known that in the case $\sigma_c < 0$, one has the representation

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |\sum_{n \geq m} a_n|}{\lambda_m}.$$

For small $\epsilon > 0$ this implies the existence of a sequence $m_j \nearrow \infty$ such that

$$|\sum_{n \geq m_j} a_n \geq e^{(\sigma_c - \epsilon)\lambda_{m_j}}$$

and we obtain (1.9) with $-\gamma = \sigma_c - \epsilon$.

The condition (BC) is very restrictive. The advantage of Theorem 1.1 is that (1.8) is always satisfied (see Section 3) for infinite number of frequencies $\lambda_{m_j - 1}$, $\lambda_{m_j}$ and the separation by $e^{-\delta \lambda_j}$ of some frequencies $\lambda_{m_j}$ only on the left is less restrictive than a separation of all frequencies on both sides. Applying Theorem 1.1, we obtain the following

**Corollary 1.1.** *Suppose $\sigma_c < 0$. Then if*

$$\liminf_{m \to \infty} \frac{\log |\sum_{n \geq m} a_n|}{\lambda_m} > -\infty, \tag{1.11}$$

*the function $\eta_D(s)$ cannot be prolonged as entire function.*

In Section 4 for $\delta > h + 2$ we construct intervals $I(\lambda_k, \delta) \subset [b, b + 1]$, $b \geq b_0$ with *clustering frequencies* and we obtain Corollary 4.1. We have infinite number of such intervals. Moreover, under some geometrical assumptions described in [PS12, Section 8] the number of such intervals is exponentially increasing when $b \to \infty$. Finally, assuming that the coefficients $a_n$ have a lower bound (4.4), we show that for every interval $I(\lambda_k, \delta)$ we have 4 possibilities concerning the behaviour of the corresponding sums. For 3 of these 4 possibilities it is possible to find frequencies satisfying (1.8), (1.9) (see Proposition 4.1).

The paper is organised as follows. In Section 2 we recall the local trace formula for $\eta_D$. Assuming $\eta_D$ entire, we deduce the estimates (2.3). This makes possible to prove that that the abscissa of $k-$ summability $\sigma_k$ of $\eta_D$ is $-\infty$. In Section 3 we prove Theorem 1.1. Section 4 is devoted to intervals $I(\lambda_k, \delta)$ with clustering frequencies and the constructions of frequencies satisfying (1.8) and (1.9).

## 2. Summation by typical means of $\eta_D$

In this section we apply the results of [DG16], [JT25, §6.1] and [Pet] for vector bundles. For our exposition we need only the local trace formula containing the poles of the meromorphic continuation of cut off resolvents $\mathbf{1}_{\tilde{V}_u}(-i\mathbf{P}_{k,\ell} - s)^{-1}\mathbf{1}_{\tilde{V}_u}$ of some operators

$$-i\mathbf{P}_{k,\ell,q},\ 0 \leq k \leq d,\ 0 \leq \ell \leq d^2 - d,\ q = 1, 2.$$

Here $\tilde{V}_u$ us a neighborhood of the trapping set $\tilde{K}_u$. The precise definitions of $\mathbf{P}_{k,\ell,q}$, $\tilde{K}_u$ and the corresponding setting are complicated and they are not necessary for the analysis below and we prefer to refer to [Pet, Section 2] for the corresponding definitions and details. Denote by $\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})$ the set of the poles of the meromorphic continuation of the corresponding cut off resolvents.

For every $A > 0$ and any $0 < \epsilon \ll 1$ we have the following local trace formula (see [Pet, Theorem 2.1])

$$\begin{aligned} &\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,2}), \mathrm{Im}\,\mu > -A}(-1)^{k+\ell}e^{-i\mu t}\\ &-\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,1}), \mathrm{Im}\,\mu > -A}(-1)^{k+\ell}e^{-i\mu t} \\ &\qquad + F_A(t) = \mathcal{F}_D(t),\ t > 0. \end{aligned} \tag{2.1}$$

Here $F_A(t) \in \mathcal{S}'(\mathbb{R})$ is supported in $(0, \infty)$, the Laplace-Fourier transform $\hat{F}_A(\lambda)$ of $F_A(t)$ is holomorphic for $\mathrm{Im}\,\lambda < A - \epsilon$ and satisfies the estimate

$$|\hat{F}_A(\lambda)| = \mathcal{O}_{A,\epsilon}(1 + |\lambda|)^{2d^2+2d-1+\epsilon},\ \mathrm{Im}\,\lambda < A - \epsilon. \tag{2.2}$$

Notice that the poles in $\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})$ are simples with positive integer residues [CP22, Theorem 1]. For the sums with fixed $q$ the cancellations in (2.1) could appear only between the terms with $k + \ell$ odd and $k+\ell$ even. On the other hand, taking the difference of sums with $q=2$ and $q=1$ we obtain more cancelations.

If the following we assume that $\eta_D$ can be a prolonged as entire function. In particular,

$$\eta_D(-i\lambda) = \langle \mathcal{F}_D, e^{it\lambda}\rangle = \sum_{\gamma\in\mathcal{P}} \frac{(-1)^{m(\gamma)}\tau^\sharp(\gamma)e^{i\lambda\tau(\gamma)}}{|\det(\operatorname{Id}-P_\gamma)|^{1/2}},\ \operatorname{Im}\lambda \geq C \gg 1$$

admits an analytic continuation for $\operatorname{Im}\lambda < C$. For fixed $A>0$ the function $\eta_D(-i\lambda)$ has no poles $\mu$ with $\operatorname{Im}\mu > -A$ and in (2.1) all terms involving poles will be canceled. Consequently, from (2.1) we obtain

$$\eta_D(-i\lambda) = \hat{F}_A(-\lambda),\ \operatorname{Im}\lambda > -A+\epsilon.$$

Setting $-i\lambda = s = \sigma + it$, $\sigma\in\mathbb{R}$, $t\in\mathbb{R}$, this implies

$$|\eta_D(s)| \leq C_A(1+|s|)^{2d^2+2d-1} \leq B_A(1+|t|)^{2d^2+2d-1},\ \sigma \geq -A+\epsilon. \tag{2.3}$$

Here we used the fact that $|\eta_D(s)|$ is bounded for $\sigma \geq C_0 > 0$ with sufficiently large $C_0>0$ and $|s| \leq \max\{A, C_0\} + |t|$ for $-A\leq\sigma\leq C_0$. We may apply the above argument for every $A>0$, so the bound (2.3) holds for every $A>0$ with constants $B_A$ depending of $A$. The crucial point is that the power $2d^2+2d-1$ is *independent* of $A$.

Applying the Phragmént- Lindelöf principle for entire function $\eta_D(s)$ in the strip

$$\{z\in\mathbb{C} : -A+\epsilon \leq \operatorname{Re} z \leq C_0\},$$

one deduces

$$|\eta_D(\sigma+it)| \leq D_{\sigma,A}(1+|t|)^{\kappa(\sigma)},\ -A+\epsilon\leq\sigma\leq C_0$$

with

$$\kappa(\sigma) = \frac{C_0-\sigma}{C_0+A-\epsilon}(2d^2+2d-1),\ -A+\epsilon\leq\sigma\leq C_0.$$

For fixed $\sigma$, taking $A$ sufficiently large we obtain for every small $0<\nu\ll 1$ the estimate

$$|\eta_D(\sigma+it)| \leq B_{\sigma,\nu}(1+|t|)^\nu,\ \sigma\leq C_0. \tag{2.4}$$

Next we recall the *summation by typical means* of Dirichlet series (see [HR64, Section IV, §2], [DS22, Section 2] for more details). For $k>0$ consider

$$C_\lambda^k(u) = \sum_{\lambda_n<u}(u-\lambda_n)^k a_n e^{-\lambda_n s}.$$

We say that the series $\sum_{n=1}^\infty a_n e^{-\lambda_n s}$ is $(\lambda, k)$ summable if

$$\lim_{u\to\infty}\frac{C_\lambda^k(u)}{u^k} = C.$$

There exists a number $\sigma_k$ such that the series is $(\lambda, k)$ summable for $\sigma > \sigma_k$ and not $(\lambda, k)$ summable for $\sigma < \sigma_k$ (see [HR64, Theorem 26]).The number $\sigma_k$ is called abscissa of $k$- summability of the series. We will apply the following

**Theorem 2.1** (Theorem 41, [HR64])**.** *Suppose that the series* $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ *admits an analytic continuation for* $\sigma > \eta$*. Suppose further that* $k$ *and* $k'$ *are positive numbers such that* $k' < k$ *and for all small* $\delta$ *we have*

$$|f(s)| \leq \mathcal{C}_\delta (1 + |s|)^{k'}$$

*uniformly for* $\sigma \geq \eta + \delta > \eta$*. Then* $f(s)$ *is* $(\lambda, k)$ *summable for* $\sigma > \eta$*.*

In fact, the above theorem in [HR64] is given without proof. The reader may consult Corollary 3.8 and Corollary 3.9 in [DS22] for a proof and other results related to Theorem 2.1 and $(\lambda, k)$ summability. The estimates (2.4) combined with Theorem 2.1 imply the following

**Proposition 2.1.** *If* $\eta_D(s)$ *can be prolonged as entire function, for every* $k > 0$ *the Dirichlet series* (1.6) *has abscissa of* $k-$*summability* $\sigma_k = -\infty$*.*

## 3. Proof of Theorem 1.1

Throughout this section we assume that $\eta_D(s)$ can be prolonged as entire function. Choose $\delta > h + 2$. First, it is easy to see that in every interval $[b, b+1], b \geq b_0 \gg 1$ we have subintervals $[\alpha, \beta] \subset [b + b + 1]$ of length greater than $e^{-\delta b}$ which does not contain frequencies. It is sufficient to write $[b, b+1]$ as an union of $e^{\delta b}$ intervals of length $e^{-\delta b}$ and to use the bounds (1.3).

We have the following simple

**Lemma 3.1.** *Fix a small* $0 < \epsilon < 1/2$*. There exists* $b_0 \geq \max\{3/h, 1\}$ *depending of* $\epsilon$ *so that for* $\alpha \geq b_0$ *we have*

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^\sharp(\gamma) \leq \alpha + \epsilon\} > \frac{\epsilon(1-\eta)e^{\alpha h}}{3(\alpha + \epsilon)}. \tag{3.1}$$

*Proof.* Choose $0 < \eta < \epsilon$ so that $4\eta \leq \frac{\epsilon h}{3(1+\epsilon)}$. For $x \geq b_0(\eta) \gg 1$ the asymptotics (1.2), imply the estimates

$$\frac{e^{hx}}{hx}(1-\eta) \leq \sharp\{\gamma \in \Pi : \tau^\sharp(\gamma) \leq x\} \leq \frac{e^{hx}}{hx}(1+\eta).$$

Therefore for $\alpha \geq b_0(\eta)$ we obtain

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^\sharp(\gamma) \leq \alpha + \epsilon\} \geq \frac{e^{h(\alpha+\epsilon)}}{h(\alpha+\epsilon)}(1-\eta) - \frac{e^{h\alpha}}{h\alpha}(1+\eta)$$

$$> \frac{(1-\eta)e^{\alpha h}}{h(\alpha+\epsilon)}\Big[1+\epsilon h - \frac{(\alpha+\epsilon)(1+\eta)}{\alpha(1-\eta)}\Big].$$

On the other hand, we have $\frac{1}{\alpha} \leq \frac{h}{3}$ and

$$4\eta \leq \frac{\epsilon h}{3(1+\epsilon)} \leq \frac{\epsilon h}{3(1+\frac{\epsilon}{\alpha})}.$$

Then

$$\frac{(\alpha+\epsilon)(1+\eta)}{\alpha(1-\eta)} = \Big(1+\frac{\epsilon}{\alpha}\Big)\Big(1+\frac{2\eta}{1-\eta}\Big) \leq (1+\frac{\epsilon}{\alpha})(1+4\eta) \leq 1+\frac{2\epsilon h}{3}$$

and one deduces (3.1). □

*Proof of Theorem 1.1.* We start with the formula for the abscissa of $k-$summability $\sigma_k < 0$ in the case when $k \in \mathbb{N}$ is an integer established by Kuniyeda [Kun16, Theorem E]. More precisely, we have

$$\sigma_k = \limsup_{u\to\infty} \frac{\log|R^k(u)|}{u^k}, \tag{3.2}$$

where $R^k(u) = \sum_{\lambda_n > u} a_n(\lambda_n - u)^k$. We are interesting of the case $k = 1$. Let $\delta, \gamma$ be the constants given in (1.8), (1.9), respectively. By Proposition 2.1, for $\eta_D(s)$ we have $\sigma_1 = -\infty$. We fix $\gamma_1 > 0$ so that $-\gamma_1 < -\delta + \gamma - 1$. Then (3.2) implies that there exists $M = M(\gamma_1) > 1$ such that

$$|R(u)| = |\sum_{\lambda_n > u} a_n(\lambda_n - u)| \leq e^{-\gamma_1 u},\ \forall u \geq M.$$

Let

$$\lambda_{m_j} - \lambda_{m_j-1} \geq Ce^{-\delta\lambda_{m_j}},\ \lambda_{m_j-2} \geq M,\ |\sum_{n\geq m_j} a_n| \geq e^{\gamma\lambda_{m_j}}.$$

Obviously, for $M$ large by using (3.1), we get $\lambda_{m_j} - \lambda_{m_j-1} < 1$.

Choose $u_{m_j-1}, u_{m_j}$ so that $\lambda_{m_j-2} < u_{m_j-1} < \lambda_{m_j-1} < u_{m_j} < \lambda_{m_j}$ and write

$$R(u_{m_j-1}) - R(u_{m_j}) = a_{m_j-1}(\lambda_{m_j-1} - u_{m_j-1}) + (u_{m_j} - u_{m_j-1}) \sum_{\lambda_n > u_{m_j}} a_n.$$

We choose $\lambda_{m_j-1} - u_{m_j-1} = \epsilon_j \ll 1$ sufficiently small to arrange

$$|a_{m_j-1}|(\lambda_{m_j-1} - u_{m_j-1}) \leq e^{-\gamma_1\lambda_{m_j}}.$$

(Exploiting (1.4), we obtain an upper bound $|a_n| \leq e^{c\lambda_n}$, $\forall n$ with $c > 0$ independent of $\lambda_n$, but this is not necessary for the estimation above.) Next

$$u_{m_j} - u_{m_j-1} = (u_{m_j} - \lambda_{m_j}) + (\lambda_{m_j} - \lambda_{m_j-1}) + (\lambda_{m_j-1} - u_{m_j-1}).$$

Taking $\epsilon_j$ very close to 0, if it is necessary, we choose $u_{m_j} = \lambda_{m_j} - \epsilon_j$, and deduce

$$u_{m_j} - u_{m_j-1} = \lambda_{m_j} - \lambda_{m_j-1} \geq Ce^{-\delta\lambda_{m_j}}.$$

Then

$$Ce^{(-\delta+\gamma)\lambda_{m_j}} \leq (u_{m_j} - u_{m_j-1})|\sum_{n\geq m_j} a_n|$$

$$= |R(u_{m_j-1}) - R(u_{m_j}) - a_{m_j-1}(\lambda_{m_j-1} - u_{m_j-1})|$$

$$\leq e^{-\gamma_1 u_{m_j-1}} + e^{-\gamma_1 u_{m_j}} + e^{-\gamma_1\lambda_{m_j}}$$

$$\leq \Big(2e^{\gamma_1(\lambda_{m_j} - u_{m_j-1})} + 1\Big)e^{-\gamma_1\lambda_{m_j}}.$$

Since

$$\lambda_{m_j} - u_{m_j-1} = \lambda_{m_j} - \lambda_{m_j-1} + \epsilon_j < 3/2,$$

the above inequality yields

$$1 \leq \frac{1}{C}\Big(2e^{\frac{3}{2}\gamma_1} + 1\Big)e^{(-\gamma_1+\delta-\gamma)\lambda_{m_j}}$$

and we obtain a contradiction for $\lambda_{m_j} \to \infty$. This completes the proof.

Since (1.8) is always satisfied for suitable frequencies $\lambda_{m_j-1}, \lambda_{m_j}$ (see Section 4), exploiting (1.11), we may arrange the condition (1.9) for $\lambda_{m_j}$ large enough. An application of Theorem 1.1 yields Corollary 1.1.

## 4. Intervals with clustering frequencies

We fix $\delta > h + 2$ and $e^{-b} < \epsilon \ll 1/2$ and consider an interval $[b, b+1]$, $b \geq b_0(\epsilon)$. Let $\lambda_k \in [b + e^{-b}, b + 1 - e^{-b}]$. To examine the clustering around $\lambda_k$, we construct some sets. Introduce

$$J_\delta(\mu) = (\mu, \mu + \mathrm{e}^{-\delta b}).$$

If $\lambda_{k+1} \notin J_\delta(\lambda_k)$, we stop the construction on the right. If $\lambda_{k+1} \in J_\delta(\lambda_k)$, one considers $J_\delta(\lambda_{k+1})$. In the case $\lambda_{k+2} \notin J_\delta(\lambda_{k+1})$, we stop the construction. Otherwise, we continue with $J_\delta(\lambda_{k+2})$ up to the situation when $\lambda_{k+q+1} \notin J_\delta(\lambda_{k+q})$. It is clear that such $q$ exists.We repeat the same construction moving on the left introducing

$$G_\delta(\mu) = (\mu - \mathrm{e}^{-\delta b}, \mu).$$

We stop when $\lambda_{k-p-1} \notin G_\delta(\lambda_{k-p})$. Set $I(\lambda_k, \delta) = [\lambda_{k-p}, \lambda_{k+q}]$. The integers $p, q$ depend on $\lambda_k$, but we omit this in the notations below. Clearly, if we take another frequency $\lambda_{k'} \in I(\lambda_k, \delta)$, we obtain by the above construction the same interval. It is not excluded that $I(\lambda_k, \delta) = \{\lambda_k\}$. In the particular case, one has $q = p = 0$. The number of the frequencies in $I(\lambda_k, \delta)$ is bounded by $e^{(h+\epsilon)(b+1)}$ and

$$\lambda_{k+q} - \lambda_{k-p} \leq e^{(h-\delta+\epsilon)b+(h+\epsilon)} < e^{-b},\ b \geq b_0(\epsilon). \tag{4.1}$$

This estimate implies $\lambda_{k+q} < b+1,\ \lambda_{k-p} > b$, so $I(\lambda_k, \delta) \subset (b, b+1)$. By Lemma 3.1, the intervals without frequencies have lengths less than $\epsilon$. Let $M(\epsilon, \delta, b)$ be the number of the sets

$$I(\lambda_k, \delta) \cup (\lambda_{k+q}, \lambda_{k+q+1}), e^{-\delta b} \leq \lambda_{k+q+1} - \lambda_{k+q} < \epsilon, \lambda_k \in [b + e^{-b}, b+1-e^{-b}].$$

Taking the union of such sets, we obtain

$$M(\epsilon, \delta, b)(\epsilon + e^{-b}) \geq 1 - 2\epsilon - 2e^{-b}.$$

For large $b$ thus implies

$$M(\epsilon, \delta, b) > \frac{1 - 2\epsilon - 2e^{-b}}{\epsilon + e^{-b}} = \frac{1}{\epsilon} - 2 + \mathcal{O}_\epsilon(e^{-b}). \tag{4.2}$$

Hence we have at least $\left[\frac{1}{\epsilon}\right] - 2$ frequencies $\lambda_{m_j} \in [b + e^{-b}, b + 1 - e^{-b}]$ with

$$\lambda_{m_j - p_j} - \lambda_{m_j - p_j - 1} > e^{-\delta \lambda_{m_j - p_j - 1}},\ \lambda_{m_j + q_j + 1} - \lambda_{m_j + q_j} > e^{-\delta \lambda_{m_j + q_j}}, \tag{4.3}$$

where $[a]$ denotes the entire part of $a$.

Now let $\gamma \gg 1$ be fixed. Given an interval $I(\lambda_k, \delta) \subset (b, b+1)$, we have 2 possibilities:

$$(i)\quad |\sum_{n \geq k-p} a_n| \geq e^{-\gamma \lambda_{k-p}},\ \ (ii)\quad |\sum_{n \geq k-p} a_n| < e^{-\gamma \lambda_{k-p}}.$$

In the case (i) the conditions (1.8), (1.9) are satisfied for $\lambda_{k-p-1}$ and $\lambda_{k-p}$. If one has (ii), and $|\sum_{n \geq k+q+1} a_n| < e^{-\gamma \lambda_{k+q+1}}$, by triangle inequality one deduces

$$|\sum_{n=k-p}^{k+q} a_n| \leq e^{-\gamma \lambda_{k-p}} + e^{-\gamma \lambda_{k+q+1}} < 2e^{-\gamma \lambda_{k-p}}.$$

Thus if (ii) holds, and $|\sum_{n=k-p}^{k+q} a_n| \geq 2e^{-\gamma \lambda_{k-p}}$ the conditions (1.8), (1.9) are satisfied for $\lambda_{k+q}$ and $\lambda_{k+q+1}$. Taking into account (4.3) and applying Theorem 1.1, we obtain the following

**Corollary 4.1.** *Suppose $\sigma_c < 0$. Suppose that there exist constants $\delta > h + 2, \gamma \gg 1$ and a sequence of intervals*

$$I(\lambda_{m_j}, \delta) = [\lambda_{m_j - p_j}, \lambda_{m_j + q_j}],\ \lambda_{m_j} \nearrow \infty$$

*satisfying (4.3) such that*

$$|\sum_{n=m_j - p_j}^{m_j + q_j} a_n| \geq 2e^{-\gamma \lambda_{m_j - p_j}}.$$

*Then $\eta_D$ cannot be prolonged as entire function.*

It is important to increase the number of intervals included in $[b, b+1]$ satisfying (4.3). By using Lemma 3.1, we see that for $\epsilon \searrow 0$ we have $b_0(\epsilon) \nearrow \infty$ so a more precise asymptotics for the counting functions of the number of frequencies with remainder is necessary. Under some geometrical assumptions, it was proved (see [PS12, Theorem 4]) that we may replace $\epsilon$ by $e^{-\mu b}$ with small $0 < \mu < h$ and obtain a lower bound of

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^\sharp(\gamma) \leq \alpha + e^{-\mu b}\}.$$

These assumptions are satisfied for $d = 2$, while for $d \geq 3$ one make some restrictions. We refer to [PS12, Section 8] for precise results and more details. Under these assumptions we obtain $M(e^{-\mu b}, \delta, b) \sim e^{\mu b}$ as $b \to \infty$ so the number of intervals satisfying (4.3) increase exponentially as $b \to +\infty$. The issue is that the possibilities to satisfy the conditions of Theorem 1.1 increase exponentially, too.

To obtain a lower bound for $|a_n|$, $\forall n \geq n_0$, introduce the condition
(L) There exist constants $c_1 > 0, c_2 > 0$, independent of $n$ such that

$$|a_n| \geq c_1 e^{-c_2 \lambda_n}, \ \forall n \geq n_0. \tag{4.4}$$

The condition (4.4) holds in the case when the lengths of primitive periodic rays $\gamma \in \Pi$ are rationally independent, because (1.7) will contain only one term and from (1.4) one deduces (4.4) with $c_1 = \min_{i \neq j} \operatorname{dist}(D_i, D_j)$ and $c_2 = d_2/2$. This rationally independence has been proved for generic domains (see [PS17, Theorem 6.2.3]). Then if (L) holds and

$$|\sum_{k \geq m} a_k| < e^{-\gamma \lambda_m}, \ |\sum_{k \geq m+1} a_k| < e^{-\gamma \lambda_{m+1}} \tag{4.5}$$

with $\gamma > c_2 + 1$, one has $c_1 e^{-c_2 \lambda_m} \leq |a_m| < 2 e^{-\gamma \lambda_m}$ which is impossible for large $\lambda_m$. Hence at least one of the estimates (4.5) does not hold.

Going back to intervals $I(\lambda_k, \delta)$, notice that for $\lambda_{k+q}$ one has also 2 possibilities:

$$(iii) \ \ |\sum_{n \geq k+q} a_n| \geq e^{-\gamma \lambda_{k+q}}, \ \ (iv) \ \ |\sum_{n \geq k+q} a_n| < e^{-\gamma \lambda_{k+q}}.$$

Assuming (L) and $\gamma > c_2 + 1$, in the case (iv) the conditions (1.8), (1.9) are satisfied for $\lambda_{k+q}$ and $\lambda_{k+q+1}$. Consequently, we obtain the following

**Proposition 4.1.** *Assume (L) satisfied and $\gamma > c_2 + 1$. Then for every interval $I(\lambda_k, \delta)$ we have 4 possibilities: $(i)-(iii), (i)-(iv), (ii)-(iii), (ii)-(iv)$. If $(i)$ holds, or if we have $(ii)-(vi)$, we may find an interval $[\lambda_{k-p-1}, \lambda_{k-p}]$ or $[\lambda_{k+q}, \lambda_{k+q+1}]$ satisfying (1.8) and (1.9).*

A more fine analysis of the estimates of the sums $|\sum_{n=k_j-p_j}^{k_j+q_j} a_n|$ should imply more precise results.

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